The Evolution of Lloyd Shapley's Game Theory in 8 Steps

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Lloyd Shapley and Al Roth won the Nobel Prize for Economic Science in 2012 for the Gale-Shapley work described below. This paper describes Game Theory as published over the career of Lloyd Shapley.

Descriptions are meant to be in the language of the layperson but at the same time reflect the underlying mathematics as developed by Shapley. References provided identify the first time that the work appeared in a public, peer-reviewed academic journal. Finally, concepts proceed from simple to complex to really complicated. These works provided the foundation for thousands of published applications over the years.

A Summary of The Work:

Lloyd Shapley provided a mathematical foundation, techniques and algorithms that enabled application of game theory to real-world situations. Lloyd contributed to the theory of 2 person and n-person games for both non-cooperative and cooperative games. His work on non-cooperative games focused on game outcomes determined by strategic choices made by individual players. His cooperative game research emphasized game outcomes determined when players form into coalitions and further identified the winning coalitions. This wide and deep range of published materials in game theory places Lloyd Shapley as the leading developer of game theory as a science.

Definitions

n players: an arbitrary number of players in a game
utility: benefit to each player for participating in the game

Utility may be in the form of relative preference, costs, product or any commodity that might be shared among the players.

transferable utility: utility in a form that can be transferred from one player to another player.
payoff: the amount of utility paid to each player before starting the game
total payoff: the total utility paid to all the players for participating before starting the game
cooperation: exchange of utility among the players with the goal of winning the game
coalition: a group of players who combine their utility to obtain the best payoff for the players in the group
outcome: a set of coalitions that allocates total payoff to the players in each coalition

coalition members share the total payoff of the game as a result of cooperation

winning: the outcome that leaves every coalition with the "best" allocation of total utility

Illustrations with these 8 Steps are on LLOYD CENTENNIAL SITE at https://harlowshapley.org/lloyd-work

1. Outcomes As Coalitions (The Shapley Value)

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!}(v(S \cup \{i\}) - v(S))$$

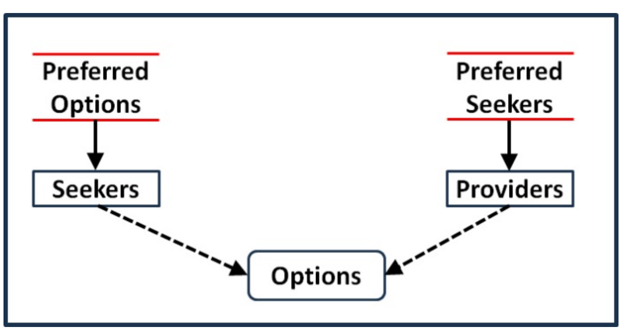
$$= \sum_{S \subseteq N \setminus \{i\}} \binom{n}{1, |S|, n-|S|-1}^{-1} (v(S \cup \{i\}) - v(S))$$

$$\varphi_i(v) = \frac{1}{n!} \sum_R \left[v(P_i^R \cup \{i\}) - v(P_i^R) \right]$$

$$\varphi_i(v) = \frac{1}{n} \sum_{S \subseteq N \setminus \{i\}} \binom{n-1}{|S|}^{-1} (v(S \cup \{i\}) - v(S))$$

An n-person game consists of n players competing to obtain a total utility payoff. The Shapley Value enables identification of **possible outcomes** in the form of coalitions whose members cooperate to share in the total utility payoff. Cooperation consists of side payments of utility among the players as an incentive to get the players to join or to quit coalitions. The Shapley Value of a player allocates the total payoff of the game among coalition members. when the player cooperates and joins a coalition. This value determines the side payment to the joining players, if any. Multiple outcomes with different groups of coalitions may form based on allocation of total payoff and the side payments. **The definition of coalitions and the benefits to cooperation represented by the Shapley Value are the keystone of the n-person game theory work contributed by Lloyd Shapley.** The Shapley Value has been used for coalition/outcome analysis on a wide range of applications, including favoritism, income taxes, control and power among stockholders of a corporation, power and stability in politics, the TN Valley Authority project cost allocation, and aircraft landing fees at Birmingham (UK) Airport.

2. Deferred Matching (Shapley Gale Matching)

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algorithm stable_matching is
    Initialize m ∈ M and w ∈ W to free
    while ∃ free man m who has a woman w to propose to do
        w := first woman on m's list to whom m has not yet proposed
        if ∃ some pair (m', w) then
            if w prefers m to m' then
                m' becomes free
                (m, w) become engaged
            end if
        else
            (m, w) become engaged
        end if
    repeat
```

A large set of people seek access to a limited number of available options from a small number of providers of those options. Providers of the options must decide which of the seekers shall be awarded access to their limited available options. Seekers state preferences regarding the options. Providers have preferences regarding the seekers that desire access to their options. The deferred matching algorithm continually varies the matches between seekers and options according to the preferences of each provider for the seeker. In each cycle of the algorithm, every remaining seeker points to the option preferred most among the remaining options. Every remaining provider points to the remaining seeker that has the highest priority for that provider. As each new seeker is considered, a previously matched seeker may be unmatched to satisfy the higher preferences for the new seeker under consideration. Unmatched persons will have to be reconsidered later in the matching process. Final matching decisions are deferred until all of the available options have been committed to a seeker without any further improvements.

This matching algorithm always guarantees that the seekers and the providers can not obtain a better result. Since the number of seekers outnumbers the number of available options, some of the seekers will not be matched with any of the limited available options. Seekers obtain the best results from this matching process by truthful reporting of preferences. This algorithm has been extensively employed over a wide range of situations, including matching college applicants (seekers) to limited university(providers) admissions(options) and matching medical school graduates(seekers) to resident programs(options) of hospitals(providers)

3. Lloyd, What Are You Thinking? (Values of N Person Games)

Type of Game	Normal-ization	Value to Player i
3-person constant-sum	A	0
	B	$\frac{1}{3}$
4-person constant-sum	A	$\frac{1}{6}\sum_{j\neq i} v(\{i,j\})$
	B	$\frac{1}{6}\sum_{j\neq i} v(\{i,j\})$
5-person constant-sum	A	$\frac{1}{10}\sum_{j\neq i} v(\{i,j\}) - \frac{1}{15}\sum_{\{j,k\}\not\ni i} v(\{j,k\})$
	B	$\frac{1}{10}\sum_{j\neq i} v(\{i,j\}) - \frac{1}{15}\sum_{\{j,k\}\not\ni i} v(\{j,k\}) + \frac{1}{5}$
2-person general-sum	A	0
	B	$\frac{1}{2}$
3-person general-sum	A	$\frac{1}{6}\sum_{j\neq i} v(\{i,j\}) - \frac{1}{3} v(I - \{i\})$
	B	$\frac{1}{6}\sum_{j\neq i} v(\{i, j\}) - \frac{1}{3} v(I - \{i\}) + \frac{1}{3}$

This example takes you directly into the mind of Lloyd Shapley as he reasons about the solutions to a game with multiple players. In this table, Lloyd Shapley demonstrates the use of inductive logic to derive his equation for the general n-person game. Inductive logic reasons from specific to general. This table begins by deriving the value to each player in a 3-person constant sum game which is a very specific game. At the end of the table, the value to each player is given for a 3 person general sum game, which is a more general game. Ultimately, these characterizations would lead to the specification of the value to each player in a general n-person game – the Shapley Value described above.

4. When The Players Are Coalitions (Shapley Value – Coalitions)

$$\varphi_C(v) = \sum_{T \subseteq N \setminus C} \frac{(n - |T| - |C|)!\, |T|!}{(n - |C| + 1)!} \sum_{S \subseteq C} (-1)^{|C|-|S|} v(S \cup T) \,.$$

$$\varphi_C(v) = \sum_{C \subseteq T \subseteq N} \frac{w(T)}{|T| - |C| + 1}$$

Players in an n-person game may themselves be fixed coalitions, each with a total utility of the coalition members. These player coalitions may combine to form super coalitions. The Shapley Value has been extended to determine **outcomes** by evaluation of super coalitions in cooperative n-coalition games. Cooperation consists of side payments among the player coalitions as an incentive to get the player coalitions to join or to quit super coalitions.

5. Who Really Wins (Core and Balanced Sets)

$$\sum_{j=1}^{p} \gamma_j v(S_j) \leqq \sum_{k=1}^{q} \delta_k v(T_k)$$

$$\gamma_1 v(S_1) + \cdots + \gamma_p v(S_p) \leqq v(N), \quad \text{all } \gamma_j > 0.$$

$$\sum_{j=1}^{p} \gamma_j v(S_j) \leq v(N),$$

$$\mathscr{T} = \begin{cases} \mathscr{S} \cup \{T\} - \{Q\} & \text{if } \gamma_Q < \gamma_R, \\ \mathscr{S} \cup \{T\} - \{Q, R\} & \text{if } \gamma_Q = \gamma_R. \end{cases}$$

$$L(\mathscr{T}, v) = L(\mathscr{S}, v) + \gamma_Q v(T) - \gamma_Q v(Q) - \gamma_Q v(R) \geq L(\mathscr{S}, v),$$

		Weights	Depth	Multiplicity
a.	$\{\overline{12}, \overline{34}\}$	1, 1	1	3
b.	$\{\overline{123}, \overline{4}\}$	1, 1	1	4
c.	$\{\overline{12}, \overline{3}, \overline{4}\}$	1, 1, 1	1	6
d.	$\{\overline{123}, \overline{124}, \overline{34}\}$	1, 1, 1	2	6
e.	$\{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$	1, 1, 1, 1	1	1
f.	$\{\overline{12}, \overline{13}, \overline{23}, \overline{4}\}$	1, 1, 1, 2	2	4
g.	$\{\overline{123}, \overline{14}, \overline{24}, \overline{3}\}$	1, 1, 1, 1	2	12
h.	$\{\overline{123}, \overline{14}, \overline{24}, \overline{34}\}$	2, 1, 1, 1	3	4
i.	$\{\overline{123}, \overline{124}, \overline{134}, \overline{234}\}$	1, 1, 1, 1	3	1
				41

Multiple possible outcomes with different coalition memberships may win the game. The Shapley Value provides those **possible outcomes**. The possible outcomes that can actually win the game is a solution to the game. The core of the game is those outcomes that cannot be improved upon by any coalition and cannot be blocked by any coalition. Outcomes that can actually win the game appear in the core. Outcomes in the core are those outcomes (based on Shapley Values) in which the players would not be better off by playing alone or by switching coalitions.

In some games, the core is empty. A core will be empty if some of the players in possible outcomes would be better off by playing alone or by switching coalitions. Possible outcomes may result in coalitions in which players are better off forming **smaller coalitions** and simply sharing the total payoff of the game. The set of possible outcomes which are better off forming smaller coalitions is called the stable set. Typically, the stable set contains more possible outcomes than the core and is more likely to lead to solutions than the more restrictive conditions for the inclusion in the core solution.

A common and important application of the core is identifying competitive prices for winning coalitions in commodity trading markets. Another common application of the core has been to find competitive prices for winning prices in the real-estate housing market.

6. A Majority Wins (Shapley-Shubik Power Index)

$$\varphi_i = (1/n!) \sum \{(|S|-1)!(n-|S|)! : S \in \mathcal{W}, S - i \notin \mathcal{W}\}$$

A simple game is one in which players/coalitions either win or lose the total payoff. For these games, Lloyd Shapley and Martin Shubik collaborated to identify a modified version of the Shapley Value. The foundation for this method is to identify pivot players. A pivot player is one who will cause a coalition to transition from losing to winning if the player joins the coalition or vice-versa. All other players cannot effect the outcome and are called dummy players. Pivot players have all the control and are given a share of the payoff in determining the outcome of the game. Dummy players have no value and do not impact the outcomes of the game.

This Shapley-Shubik power index has the greatest application in games where a majority wins. In their initial paper, the indexing method was applied to decision making in a committee. Other applications have included the electoral college presidential elections, shareholder control of corporations and corporate buyouts, and limits imposed on small parties in Italy.

7. Sharing The Costs (Aumann – Shapley Value)

$$(Sv)(ds) = \int_0^1 (\, v(tI + ds) - v(tI)\,)\, dt.$$

$$\mu(tI) = t\mu(I),$$

$$v(tI + ds) = f(t\mu(I)) + f'(t\mu(I))\mu(ds).$$

$$(Sv)(ds) = \int_0^1 f'_{t\mu(I)}(\mu(ds))\, dt$$

$$(Sv)(ds) = \lim_{\varepsilon \to 0, \varepsilon > 0} \frac{1}{\varepsilon} \int_0^{1-\varepsilon} (f(t + \varepsilon\mu(ds)) - f(t))\, dt$$

Many games have a large number of players. So many players that any individual player cannot exert any appreciable influence on the outcome of the game. These games are called non-atomic games.

A group of customers requires a service or a product. This service or product is only available in bulk in greater amounts than is required by any individual customer. Alternatively, purchase of the service or product may have a very high initial fee that cannot be covered by an individual customer. If the customers form coalitions, the combined demand of the coalition may make the acquisition feasible. Costs must be shared across the coalition members in a fair manner.

Each use of the service or the product is considered a player. While the number of customers may be small, the number of users within each customer is large. Each user employs the service for a short period of time. Each short time period is now a player in an n-person game where the number of players is extremely large. Each player has a specific characteristic or feature. All of the players are then collected into coalitions.

As with any n-person game, the outcomes are created from the combinations of coalitions. A solution to this game determines the allocation of the total charge for the service or product to each player (short time period) in each coalition. A version of the Shapley Value for a large number of players who are members of a coalition calculates the total cost to each coalition in an outcome using the rate of change of the costs of the service. This cost can then be allocated to the individual members (time periods) of the coalition. As with any n- person game, those outcomes in the core and the stable set are the solutions to the game. Costs allocated to each coalition are then passed to the users (short time periods).

This value has been applied to resolving telephone billing rates among users with shared costs, electricity rates, evaluating insurance and portfolio risks and determining shipping schedules of transportation of goods from multiple origins to multiple destinations with fixed transportation costs

8. A Roll of the Dice (Stochastic Games)

$$M + (1-s)M + (1-s)^2M + \ldots = M/s.$$

$$P^{kl} = (p_{ij}^{kl} | i = 1, 2, \ldots, m_k;\ j = 1, 2, \ldots, n_k)$$
$$A^k = (a_{ij}^{k} | i = 1, 2, \ldots, m_k;\ j = 1, 2, \ldots, n_k),$$

$$p_{ij}^{kl} \geq 0,\ |a_{ij}^k| \leq M,\ \sum_{l=1}^{N} p_{ij}^{kl} = 1 - s_{ij}^k \leq 1 - s < 1.$$

$$\mathfrak{H}^k(\vec{x}, \vec{y}) = x^k A^k y^k + \sum_l x^k P^{kl} y^k \mathfrak{H}^l(\vec{x}, \vec{y}),$$

$$\mathfrak{H}^k(\vec{x}, \vec{y}) = \frac{\begin{vmatrix} x^1P^{11}y^1 - 1 & x^1P^{12}y^1 & \ldots & -x^1A^1y^1 & \ldots & x^1P^{1N}y^1 \\ x^2P^{21}y^2 & x^2P^{22}y^2 - 1 & & & & \\ \ldots & & & \ldots & & \ldots \\ x^NP^{N1}y^N & \ldots & \ldots & -x^NA^Ny^N & \ldots & x^NP^{NN}y^N - 1 \end{vmatrix}}{\begin{vmatrix} x^1P^{11}y^1 - 1 & x^1P^{12}y^1 & \ldots & x^1P^{1k}y^1 & \ldots & x^1P^{1N}y^1 \\ x^2P^{21}y^2 & x^2P^{22}y^2 - 1 & & \ldots & & \\ \ldots & & & x^kP^{kk}y^k - 1 & & \ldots \\ & & & \ldots & & \\ x^NP^{N1}y^N & \ldots & \ldots & x^NP^{Nk}y^N & \ldots & x^NP^{NN}y^N - 1 \end{vmatrix}}.$$

$$\mathfrak{R}^k(\vec{i}) = \frac{\begin{vmatrix} p_{i_1}^{11} - 1 & p_{i_1}^{12} & \ldots & -a_{i_1}^1 & \ldots & p_{i_1}^{1N} \\ p_{i_2}^{21} & p_{i_2}^{22} - 1 & & & & \\ \ldots & & & \ldots & & \ldots \\ p_{i_N}^{N1} & & \ldots & -a_{i_N}^N & \ldots & p_{i_N}^{NN} - 1 \end{vmatrix}}{\begin{vmatrix} p_{i_1}^{11} - 1 & p_{i_1}^{12} & \ldots & p_{i_1}^{1k} & \ldots & p_{i_1}^{1N} \\ p_{i_2}^{21} & p_{i_2}^{22} - 1 & & \ldots & & \\ \ldots & & & p_{i_k}^{kk} - 1 & & \ldots \\ & & & \ldots & & \\ p_{i_N}^{N1} & & \ldots & p_{i_N}^{Nk} & \ldots & p_{i_N}^{NN} - 1 \end{vmatrix}}.$$

Two players are attempting to control an environment. This environment exists in one of several states. As play proceeds, the players make decisions that alter the state of the environment. One player receives a payoff from the other player. The game may continue or terminate after each play. A maximum total payoff may be earned over the life of the game. State changes in the environment, payoff transfers and continuation/termination decisions at each play are controlled by probabilities. These probabilities change as play proceeds.

At each play of the game, the player faces a contradiction. Payoff from his decision contributes to his current total payoff but may result in termination of the game. As a result, the player may accrue a smaller total payoff despite maximizing the payoff for each play of the game until termination.

A solution for this type of game consists of a strategy that provides a decision for each player at every possible play of the game for every possible history of decisions. Shapley proved that such a solution exists. Moreover, he proved that these games have a value that represents the worth of the game to the players when future possible payoffs are discounted at a constant positive rate.

Using his theories, a method is defined that enables generation of all of the strategies for each history of play at each play of the game. As the game is played, players can use these generated strategies to play the game, dependent on the exact history at the play of the game.

Stochastic games have been used for power management in cognitive networks, for human machine dialogs and for a wide range of applications.

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